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On the stability index for weighted composition operators

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Abstract

Let $\epsilon > 0$. A continuous linear operator $T : C(X) \longrightarrow C(Y)$ is said to *ϵ-preserve disjointness* if $\|(Tf)(Tg)\|_\infty \leq \epsilon$, whenever $f, g \in C(X)$ satisfy $\|f\|_\infty = \|g\|_\infty = 1$ and $fg \equiv 0$. In this paper we continue our study of the minimal interval where the possible maximal distance from a norm one operator which ϵ-preserves disjointness to the set of weighted composition maps may lie. We provide sharp bounds for both the finite and the infinite case, which turn out to be completely different.

Keywords: Instability; ϵ-disjointness preserving; Weighted composition operator; Spaces of continuous functions

1. Introduction

Let $\mathbb{K}$ denote the field of real or complex numbers. Let $C(X)$ stand for the Banach space of all $\mathbb{K}$-valued continuous functions defined on a compact Hausdorff space X and equipped with its usual supremum norm.

An operator $S : C(X) \longrightarrow C(Y)$ is said to be a *weighted composition map* if there exist a function $a \in C(Y)$ and a map $h : Y \longrightarrow X$ such that h is continuous on $c(a) :=$

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$\{y \in Y : a(y) \neq 0\}$ and

$$(Sf)(y) = a(y)f(h(y))$$

for every $f \in C(X)$ and $y \in Y$. In particular, the zero operator is a weighted composition map ($a \equiv 0$ and h a constant map).

Obviously every weighted composition map is linear and continuous, and is also *disjointness preserving*, in the sense that given $f, g \in C(X)$, $fg \equiv 0$ yields $(Sf)(Sg) \equiv 0$. Conversely, it is well known that a *continuous* disjointness preserving operator is a weighted composition (see for instance [6,5,7]).

Given $\epsilon > 0$, a continuous linear operator $T : C(X) \longrightarrow C(Y)$ is said to *ϵ-preserve disjointness* if $\|(Tf)(Tg)\|_\infty \leq \epsilon$, whenever $f, g \in C(X)$ satisfy $\|f\|_\infty = \|g\|_\infty = 1$ and $fg \equiv 0$ (or, equivalently, if $\|(Tf)(Tg)\|_\infty \leq \epsilon \|f\|_\infty \|g\|_\infty$ whenever $fg \equiv 0$).

We denote by $\epsilon - \mathbf{DP}(X, Y)$ the set of all norm one operators ϵ-preserving disjointness from $C(X)$ to $C(Y)$, and by $\mathbf{WCM}(X, Y)$ the set of all weighted composition maps from $C(X)$ to $C(Y)$.

We are interested in the *deviation* of $\epsilon - \mathbf{DP}(X, Y)$ from $\mathbf{WCM}(X, \mathbb{K})$, that is, the possible maximal distance from a norm one operator which ϵ-preserves disjointness to the set of weighted composition maps. Given $\epsilon > 0$ and two compact Hausdorff spaces, X and Y, we denote this deviation by $\mathbf{S}(X, Y)(\epsilon)$, that is,

$$\mathbf{S}(X, Y)(\epsilon) := \sup\{\mathrm{dist}(T, \mathbf{WCM}(X, Y)) : T \in \epsilon - \mathbf{DP}(X, Y)\}$$

where $\mathrm{dist}(T, \mathbf{WCM}(X, Y)) = \inf\{\|T - S\| : S \in \mathbf{WCM}(X, Y)\}$.

Since the zero operator is a weighted composition map, it is obvious that $\mathbf{S}(X, Y)(\epsilon) \leq 1$. In [4] Dolinar proved that $\mathbf{S}(X, Y)(\epsilon) \leq 20\sqrt{\epsilon}$. This bound was recently sharpened to $\sqrt{17\epsilon/2}$ in [1], where it was also proved, by means of an example of X, Y and T (for each $\epsilon < 2/17$), that this new bound cannot be improved.

In this paper we pursue our study of the *stability index* $\mathbf{S}(X, Y)(\epsilon)$, trying to find, for a given ϵ, the *minimal* interval where this index may lie. We prove that the lower endpoint of that interval does not depend on the topological features of the space X but on its cardinality (denoted by card X). If X is infinite, then

$$\mathbf{S}(X, Y)(\epsilon) \geq 2\sqrt{\epsilon},$$

and this value is attained for all X and some Y (see Theorems 2.1 and 2.2). On the contrary, if X is finite, then

$$\mathbf{S}(X, Y)(\epsilon) \leq 2\sqrt{\epsilon},$$

and the stability index may take only *two* values (see Theorem 2.4). A question arises naturally at this point. Can we find two spaces X and Y for which $\mathbf{S}(X, Y)(\epsilon)$ lies strictly between the bounds $2\sqrt{\epsilon}$ and $\sqrt{17\epsilon/2}$? We shed some light on this question in Theorem 2.3 and Example 5.2, where we prove that there exists Y for which $\mathbf{S}(X, Y)(\epsilon) = \sqrt{8\epsilon}$ whenever X is the one-point compactification of any infinite discrete space. Notice, in particular, the big difference between the finite case and the simplest infinite case (that is, when $X = \mathbb{N} \cup \{\infty\}$).

The organization of the paper is as follows. In Section 2 we present the main results of the paper without proofs. In Section 3 we focus on the results related to the lower endpoint of the above mentioned interval, whereas in Section 4 the emphasis is on the upper bound. Section 5 contains the example mentioned in the former paragraph. Finally, Section 6 consists of the proofs of the main results.

Notation. Throughout the paper, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Furthermore X and Y will be compact Hausdorff spaces with at least two points (when X has just one point we obtain a trivial case, and when Y consists of a single point, we are dealing with functionals, and the results take a completely different form, as can be seen in [2]).

Given a compact Hausdorff space Z, $C(Z)'$ will denote the space of linear and continuous functionals defined on $C(Z)$. For $\varphi \in C(Z)'$, we will write λ_φ to denote the measure which represents it. Also, for $x \in Z$, δ_x will be the evaluation functional at x, that is, $\delta_x(f) := f(x)$ for every $f \in C(Z)$, and given $T : C(X) \longrightarrow C(Y)$ linear and continuous, we set $T_y := \delta_y \circ T$ for each $y \in Y$. A functional $\varphi \in C(Z)'$ is said to *ϵ-preserve disjointness* if $|\varphi(f)\varphi(g)| \leq \epsilon$, whenever $f, g \in C(Z)$ satisfy $\|f\|_\infty = \|g\|_\infty = 1$ and $fg \equiv 0$. The set of all (not necessarily of norm one) functionals on $C(Z)$ ϵ-preserving disjointness will be denoted by $\epsilon - \mathbf{DP}(Z, \mathbb{K})$.

For $f \in C(Z)$ and $r > 0$, $0 \leq f \leq r$ means that $f(x) \in [0, r]$ for every $x \in Z$, $c(f) = \{x \in Z : f(x) \neq 0\}$ denotes its cozero set and $\operatorname{supp}(f)$ its support. Also $\mathbf{1}$ will be the constant function equal to 1 and, for $A \subset Z$, ξ_A will be the characteristic function of A.

In a Banach space E, for $\mathrm{e} \in E$ and $r > 0$, $B(\mathrm{e}, r)$ and $\overline{B}(\mathrm{e}, r)$ denote the open and the closed ball with center e and radius r, respectively.

2. Main results

In this section we present the main results of the paper. For their proofs (which can be found in Section 6), we need results given in Sections 3–5.

Theorem 2.1. *Let $\epsilon > 0$. If X is infinite, then*

$$\min\left\{2\sqrt{\epsilon}, 1\right\} \leq \mathbf{S}(X, Y)(\epsilon) \leq \min\left\{\sqrt{\frac{17\epsilon}{2}}, 1\right\}.$$

In the following theorem, we see that the above lower bounds are sharp for some families of extremely disconnected spaces Y. This should be compared with [1, Example 4.6], where the local connectedness of some other spaces Y plays an important rôle when proving that their corresponding upper bounds are sharp.

Theorem 2.2. *Let $\epsilon > 0$. If X is infinite and Y is the Stone–Čech compactification of a discrete space, then*

$$\mathbf{S}(X, Y)(\epsilon) = \min\left\{2\sqrt{\epsilon}, 1\right\}.$$

Next we study the existence of compact Hausdorff spaces X and Y for which $\mathbf{S}(X, Y)(\epsilon)$ satisfies the strict inequalities

$$\min\left\{2\sqrt{\epsilon}, 1\right\} < \mathbf{S}(X, Y)(\epsilon) < \min\left\{\sqrt{\frac{17\epsilon}{2}}, 1\right\}.$$

The following result and Example 5.2 provide an affirmative answer to this question.

Theorem 2.3. *Let $\epsilon > 0$. If X is the one-point compactification of an infinite discrete space, then*

$$\mathbf{S}(X, Y)(\epsilon) \leq \min\left\{\sqrt{8\epsilon}, 1\right\}.$$

In the finite case, the conclusions are very different. For $\epsilon < 1/4$, the stability index may take only two values, one of them being $2\sqrt{\epsilon}$, and the other being given in terms of the function $r_X : (0, +\infty) \longrightarrow \mathbb{R}$ (recall that we are assuming card $X \geq 2$), defined by

$$r_X(\epsilon) := \begin{cases} \left(1 - \dfrac{1}{n}\right) \min \left\{ \sqrt{\dfrac{n^2}{n^2 - 1}} 2\sqrt{\epsilon}, 1 \right\} & \text{if } n := \operatorname{card} X \text{ is odd} \\ \left(1 - \dfrac{1}{n}\right) 2\sqrt{\epsilon} & \text{if } n := \operatorname{card} X \text{ is even.} \end{cases}$$

Theorem 2.4. *Let $\epsilon > 0$. Suppose that X is finite. If Y is zero dimensional, then* $\mathbf{S}(X, Y)(\epsilon) = \min \{r_X(\epsilon), 1\}$*; otherwise* $\mathbf{S}(X, Y)(\epsilon) = \min \left\{2\sqrt{\epsilon}, 1\right\}$.

3. The lower bound

Lemma 3.1. *Let μ be a regular Borel probability measure on X, and let $y_0 \in Y$. If $0 < r \leq 1$, then there exists a continuous linear operator $T : C(X) \longrightarrow C(Y)$, $\|T\| = 1$, with the following two properties:*

(1) *For every $f, g \in C(X)$ with $\|f\|_\infty = 1 = \|g\|_\infty$ and $fg \equiv 0$,*

$$\|(Tf)(Tg)\|_\infty \leq r^2 \mu(c(f))(1 - \mu(c(f))).$$

(2) *Given $S \in \mathbf{WCM}(X, Y)$ with the associated map $h : Y \longrightarrow X$,*

$$\|T - S\| \geq r(1 - \mu(\{h(y_0)\})).$$

Proof. Fix $x_0 \in X$ and $y_1 \in Y$, $y_1 \neq y_0$. After choosing two disjoint neighborhoods, $U(y_0)$ and $U(y_1)$, of y_0 and y_1, respectively, we consider two functions $\alpha, \beta \in C(Y)$ with the following properties:

- $0 \leq \alpha \leq r$, $0 \leq \beta \leq 1$,
- $\alpha(y_0) = r$, $\operatorname{supp}(\alpha) \subset U(y_0)$,
- $\beta(y_1) = 1$, $\operatorname{supp}(\beta) \subset U(y_1)$.

Define $T : C(X) \longrightarrow C(Y)$ by

$$(Tf)(y) := \alpha(y) \int_X f \mathrm{d}\mu + \beta(y)\delta_{x_0}(f)$$

for every $f \in C(X)$ and $y \in Y$. It is easy to check that T is linear and continuous, and that $\|T\| = 1$.

Let $f, g \in C(X)$ with $\|f\|_\infty = \|g\|_\infty = 1$ and $fg \equiv 0$. It is clear that, for each $y \in Y$,

$$\begin{aligned} |(Tf)(y)(Tg)(y)| &= \left| \left(\alpha(y) \int_X f \mathrm{d}\mu\right) \left(\alpha(y) \int_X g \mathrm{d}\mu\right) \right| \\ &= \alpha(y)^2 \left| \int_X f \mathrm{d}\mu \right| \left| \int_X g \mathrm{d}\mu \right| \\ &\leq \alpha(y)^2 \mu(c(f)) \mu(c(g)) \\ &\leq r^2 \mu(c(f))(1 - \mu(c(f))). \end{aligned}$$

On the other hand, let $S \in \mathbf{WCM}(X, Y)$ with the associated map $h : Y \longrightarrow X$. It is clear that, if $(S\mathbf{1})(y_0) = 0$, then $\|T - S\| = |(T - S)(\mathbf{1})(y_0)| \geq r$. If $y_0 \in c(S\mathbf{1})$, and U is an open neighborhood of $h(y_0)$, then select $f \in C(X)$ satisfying $0 \leq f \leq 1$, $f(h(y_0)) = 0$, and $f \equiv 1$ on $X \setminus U$. Obviously $(Sf)(y_0) = 0$ and $|(Tf)(y_0)| = \alpha(y_0) \int_X f \mathrm{d}\mu$. Hence

$$\begin{aligned}\|T - S\| &\geq |(Tf)(y_0)| \\ &\geq \alpha(y_0) \int_{X \setminus U} f \mathrm{d}\mu \\ &\geq r(1 - \mu(U)).\end{aligned}$$

The conclusion follows from the regularity of μ. $\square$

The following result depends on whether or not the space X admits a *continuous* measure (recall that a *Borel* measure on a Hausdorff space is said to be continuous if it vanishes on all singletons; see for instance [3, Definition 7.14.14]).

Corollary 3.2. *Let $0 < \epsilon < 1/4$. Suppose that X is infinite. Then for each $t < 1$, there exists $T \in \epsilon - \mathbf{DP}(X, Y)$ such that*

$$B\left(T, 2t\sqrt{\epsilon}\right) \cap \mathbf{WCM}(X, Y) = \emptyset.$$

Furthermore, if X admits a continuous regular probability measure, then T can be taken such that

$$B\left(T, 2\sqrt{\epsilon}\right) \cap \mathbf{WCM}(X, Y) = \emptyset.$$

Proof. We apply Lemma 3.1, where we consider $r = 2\sqrt{\epsilon}$. In the first part, we take a regular Borel probability measure μ on X such that $\mu(\{x\}) < \delta$ for every $x \in X$ (for $\delta := 1 - t$), whereas in the second part, the measure of each point being zero, δ can be taken as small as required. $\square$

Corollary 3.3. *Let $0 < \epsilon < 1/4$. Suppose that X is finite. Then there exists $T \in \epsilon - \mathbf{DP}(X, Y)$ such that*

$$B\left(T, r_X(\epsilon)\right) \cap \mathbf{WCM}(X, Y) = \emptyset.$$

Proof. For $n := \operatorname{card} X$, consider the measure μ on X such that $\mu(\{x\}) = 1/n$ for every $x \in X$. In particular, if n is odd and $E \subset X$,

$$\mu(E)(1 - \mu(E)) \leq \frac{n-1}{2n}\frac{n+1}{2n} = \frac{n^2 - 1}{4n^2}.$$

We take $r = 2\sqrt{\epsilon}$ if n is even, and $r = \min\left\{2n\sqrt{\epsilon/(n^2 - 1)}, 1\right\}$ if n is odd. The conclusion follows from Lemma 3.1. $\square$

We next show that the bound provided in Corollary 3.2 is sharp whenever Y consists of the Stone–Čech compactification of any discrete space. We need the following result, which will be also used later.

Lemma 3.4. *Let $0 < \epsilon < 1/4$. The function $\gamma : \left[2\sqrt{\epsilon}, 1\right] \longrightarrow \mathbb{R}$, defined by $\gamma(t) := t - \sqrt{t^2 - 4\epsilon}$ is strictly decreasing and bounded above by $2\sqrt{\epsilon}$.*

Proposition 3.5. *Let* $0 < \epsilon < 1/4$. *Suppose that* Y *is the Stone–Čech compactification of a discrete space, and that* X *is infinite. Let* $T \in \epsilon - \mathbf{DP}(X, Y)$. *Then*

$$\overline{B}\left(T, 2\sqrt{\epsilon}\right) \cap \mathbf{WCM}(X, Y) \neq \emptyset.$$

Furthermore, if X *does not admit a continuous regular probability measure and* Y *is finite, then*

$$B\left(T, 2\sqrt{\epsilon}\right) \cap \mathbf{WCM}(X, Y) \neq \emptyset.$$

Proof. Let Z be a discrete space such that $Y = \beta Z$. Of course Y may be finite (that is, $Y = Z$), and this is necessarily the case when we consider the second part of the theorem. Let $Z_0 := \left\{y \in Z : \left\|T_y\right\| > 2\sqrt{\epsilon}\right\}$, which is a nonempty closed and open subset of Z, and let

$$Z_1 := \left\{z \in Z \setminus Z_0 : \exists x_z \in X \text{ with } \left|\lambda_{T_z}(\{x_z\})\right| > 0\right\}.$$

Fix any $x_0 \in X$. By [1, Lemma 2.3], we can define a map $h : Z \longrightarrow X$ such that $\left|\lambda_{T_z}(\{h(z)\})\right| \geq \sqrt{\|T_z\|^2 - 4\epsilon}$ for every $z \in Z_0$, and such that $h(z) := x_z$ for $z \in Z_1$, and $h(z) := x_0$ for $z \notin Z_0 \cup Z_1$. Also, since Z is discrete, h is continuous, and consequently it can be extended to a continuous map from Y to X (when $Y \neq Z$). We will denote this extension also by h.

Define $\alpha : Z \longrightarrow \mathbb{K}$ by $\alpha(z) := \lambda_{T_z}(\{h(z)\})$ if $z \in Z_0 \cup Z_1$, and $\alpha(z) := 0$ otherwise, and extend it to a continuous function, also called α, defined on Y. Then consider $S : C(X) \longrightarrow C(Y)$ defined by $(Sf)(y) := \alpha(y) f(h(y))$ for every $f \in C(X)$ and $y \in Y$.

Let us check that $\|T - S\| \leq 2\sqrt{\epsilon}$. Take $f \in C(X)$ with $\|f\|_\infty \leq 1$. First, suppose that $z \in Z \setminus (Z_0 \cup Z_1)$. Then $(Sf)(z) = 0$, so

$$|(Tf)(z) - (Sf)(z)| = |(Tf)(z)| \leq 2\sqrt{\epsilon}.$$

Now, if $z \in Z_1$, then $\|T_z\| \leq 2\sqrt{\epsilon}$ and, as in the proof of [1, Lemma 2.4],

$$|(Tf)(z) - (Sf)(z)| \leq \|T_z\| - \left|\lambda_{T_z}(\{h(z)\})\right| < 2\sqrt{\epsilon}.$$

On the other hand, if $z \in Z_0$, we know by [1, Corollary 2.5] that

$$|(Tf)(z) - (Sf)(z)| \leq \|T_z\| - \sqrt{\|T_z\|^2 - 4\epsilon}.$$

By Lemma 3.4, $|(Tf)(z) - (Sf)(z)| < 2\sqrt{\epsilon}$ for every $z \in Z_0$. By continuity, we see that the same bound applies to every point in Y, and the first part is proved.

Finally, in the second case, that is, when X does not admit a continuous regular probability measure and Y is finite, $Y = Z$, and $Z \setminus (Z_0 \cup Z_1)$ consists of those points satisfying $\|T_z\| = 0$. The conclusion is then easy. □

In what follows, we shall need the next result [2, Proposition 3.4].

Proposition 3.6. *Let* $0 < \epsilon < 1/4$. *Suppose that* X *is a finite set of cardinality* $k \in 2\mathbb{N}$. *If* $\varphi \in \epsilon - \mathbf{DP}(X, \mathbb{K})$ *and* $\|\varphi\| = 1$, *then there exists* $x \in X$ *such that*

$$\left|\lambda_\varphi(\{x\})\right| \geq \frac{1 + \sqrt{1 - 4\epsilon}}{k}.$$

Proposition 3.7. *Let* $0 < \epsilon < 1/4$. *Suppose that* X *is finite and that* $\varphi \in \epsilon - \mathbf{DP}(X, \mathbb{K})$ *satisfies* $\|\varphi\| \leq 1$. *Then there exists* $x \in X$ *such that*

$$\left\|\varphi - \lambda_\varphi(\{x\})\delta_x\right\| \leq r_X(\epsilon).$$

Proof. Assuming that $X := \{x_1, \ldots, x_n\}$, we can choose a point $x \in X$ such that $\left|\lambda_\varphi(\{x\})\right| \geq \left|\lambda_\varphi(\{x_i\})\right|$ for every $x_i \in X$, which yields $\left|\lambda_\varphi(\{x\})\right| \geq \|\varphi\|/n$.

Fix any $f \in C(X)$, $\|f\|_\infty \leq 1$. Consequently $\left|\varphi(f) - \lambda_\varphi(\{x\})\delta_x(f)\right| \leq (n-1)\|\varphi\|/n$ and, if $\|\varphi\| \leq 2\sqrt{\epsilon}$, then

$$\left|\varphi(f) - \lambda_\varphi(\{x\})\delta_x(f)\right| \leq \frac{2(n-1)}{n}\sqrt{\epsilon} \leq r_X(\epsilon).$$

Let us now study the case when $\|\varphi\| > 2\sqrt{\epsilon}$. We know from [1, Corollary 2.5] that $\left|\varphi(f) - \lambda_\varphi(\{x\})\delta_x(f)\right| \leq \|\varphi\| - \sqrt{\|\varphi\|^2 - 4\epsilon} = \gamma(\|\varphi\|)$ (see also Lemma 3.4). Next, we split the proof into two cases.

Case 1. *Suppose that n is odd*. We see that, to finish the proof in this case, it is enough to show that

$$m_1(\|\varphi\|) := \min\left\{\gamma(\|\varphi\|), \frac{n-1}{n}\|\varphi\|\right\} \leq r_X(\epsilon). \tag{3.1}$$

To do this, we consider the function $\delta : \left(2\sqrt{\epsilon}, 1\right] \longrightarrow \mathbb{R}$ defined by $\delta(t) := (n-1)t/n$ for every t.

If $1 \leq t_0 := 2n\sqrt{\epsilon/(n^2-1)}$, then $r_X(\epsilon) = (n-1)/n$, and the inequality (3.1) is obvious. The other possibility is that $t_0 < 1$, implying that $r_X(\epsilon) = 2\sqrt{\epsilon(n-1)/(n+1)}$. In this case we see that $r_X(\epsilon) = \delta(t_0) = \gamma(t_0)$. Also δ is increasing in $\left(2\sqrt{\epsilon}, t_0\right]$, so we deduce that $m_1(\|\varphi\|) = \delta(\|\varphi\|)$ if $2\sqrt{\epsilon} < \|\varphi\| \leq t_0$, and $m_1(\|\varphi\|) = \gamma(\|\varphi\|)$ otherwise. We conclude that in every case $m_1(\|\varphi\|) \leq r_X(\epsilon)$.

Case 2. *Suppose that n is even*. Let $\eta : \left[2\sqrt{\epsilon}, 1\right] \longrightarrow \mathbb{R}$ be defined by

$$\eta(t) := t - \frac{t + \sqrt{t^2 - 4\epsilon}}{n}$$

for every $t \in \left[2\sqrt{\epsilon}, 1\right]$. By Proposition 3.6, it follows that

$$\left|\lambda_\varphi(\{x\})\right| \geq \left(\|\varphi\| + \sqrt{\|\varphi\|^2 - 4\epsilon}\right)\Big/ n,$$

so

$$\left|\varphi(f) - \lambda_\varphi(\{x\})\delta_x(f)\right| \leq \eta(\|\varphi\|).$$

Consequently, to finish the proof we just need to show that

$$m_2(\|\varphi\|) := \min\{\gamma(\|\varphi\|), \eta(\|\varphi\|)\} \leq \frac{2(n-1)\sqrt{\epsilon}}{n} = r_X(\epsilon).$$

It is clear that, when $n = 2$, $\eta = \gamma/2$, and the above inequality follows from Lemma 3.4. So we assume that $n \neq 2$, and see that $\eta(t) \leq \gamma(t)$ if and only if $t \in \left[2\sqrt{\epsilon}, A\right]$, for $A := \min\left\{1, 2(n-1)\sqrt{\epsilon/(n^2-2n)}\right\}$. Since η is decreasing in $\left[2\sqrt{\epsilon}, A\right]$, we deduce that

$$m_2(\|\varphi\|) \leq \eta\left(2\sqrt{\epsilon}\right) = \frac{2(n-1)\sqrt{\epsilon}}{n},$$

as was required. $\square$

Remark 3.1. If X is finite and $T : C(X) \longrightarrow C(Y)$ is linear and continuous, then for $C \subset X$ nonempty, $\lambda_{T_y}(C) = \sum_{x\in C} \left(T\xi_{\{x\}}\right)(y)$ and $\left|\lambda_{T_y}\right|(C) = \sum_{x\in C} \left|\left(T\xi_{\{x\}}\right)(y)\right|$ for every $y \in Y$, and consequently the maps from Y to $\mathbb{K}$ given by $y \mapsto \lambda_{T_y}(C)$ and $y \mapsto \left|\lambda_{T_y}\right|(C)$ are continuous. In particular $y \mapsto \left\|T_y\right\| = \left|\lambda_{T_y}\right|(X)$ is continuous.

Corollary 3.8. *Let* $0 < \epsilon < 1/4$. *Suppose that* Y *is the Stone–Čech compactification of a discrete space, and that* X *is finite. Let* $T \in \epsilon - \mathbf{DP}(X, Y)$. *Then*

$$\overline{B}\left(T, r_X(\epsilon)\right) \cap \mathbf{WCM}(X, Y) \neq \emptyset.$$

Proof. Suppose that Z is a discrete space with $Y = \beta Z$. By Proposition 3.7, there exists a map $h : Z \longrightarrow X$ such that

$$\left\|T_z - \lambda_{T_z}(\{h(z)\})\delta_{h(z)}\right\| \le r_X(\epsilon)$$

for every $z \in Z$. If we now define $\alpha : Z \longrightarrow \mathbb{K}$ by $\alpha(z) := \lambda_{T_z}(\{h(z)\})$ for each $z \in Z$, then both h and α can be extended to a continuous function defined on the whole Y (when Z is infinite). We denote these extensions also by h and α, respectively. We define $S : C(X) \longrightarrow C(Y)$ by $(Sf)(y) := \alpha(y) f(h(y))$ for every $f \in C(X)$ and $y \in Y$.

Finally, by denseness of Z in Y, we conclude that $\|T - S\| \le r_X(\epsilon)$. $\square$

Corollary 3.9. *Let* $0 < \epsilon < 1/4$. *Suppose that* Y *is zero dimensional, and that* X *is finite. Let* $\delta > 0$ *and* $T \in \epsilon - \mathbf{DP}(X, Y)$. *Then*

$$B\left(T, r_X(\epsilon) + \delta\right) \cap \mathbf{WCM}(X, Y) \neq \emptyset.$$

Proof. For each $y \in Y$, let $M_y := \max\left\{\left|\lambda_{T_y}(x)\right| : x \in X\right\}$. Consider $x_1 \in X$ such that the set $K_1 := \left\{y \in Y : \left|\lambda_{T_y}(x_1)\right| = M_y\right\}$ is nonempty. Clearly K_1 is compact because it coincides with the set of all $y \in Y$ satisfying $\left|T\xi_{\{x_1\}}(y)\right| \ge \left|T\xi_{\{x\}}(y)\right|$ for $x \neq x_1$. Taking into account Proposition 3.7, $\left\|T_y - \lambda_{T_y}(\{x_1\})\delta_{x_1}\right\| \le r_X(\epsilon)$, so by continuity (see Remark 3.1) there exists a closed and open neighborhood $U(y)$ of y such that, for every $z \in U(y)$,

$$\left\|T_z - \lambda_{T_z}(\{x_1\})\delta_{x_1}\right\| < r_X(\epsilon) + \delta.$$

By compactness, we conclude that there exist $y_1, \ldots, y_{n_1}$ with $K_1 \subset L_1 := \bigcup_{i=1}^{n_1} U(y_i)$. Next we take $x_2 \in X$ such that the set $K_2 := \left\{y \in Y \setminus L_1 : \left|\lambda_{T_y}(x_2)\right| = M_y\right\}$ is nonempty, and proceed in a similar way as above to obtain a closed and open set L_2 with $K_2 \subset L_2$ such that $\left\|T_z - \lambda_{T_z}(\{x_2\})\delta_{x_2}\right\| < r_X(\epsilon) + \delta$ for every $z \in L_2$.

After a finite number of steps we finish this process, obtaining points x_k and closed and open sets L_k. We define $h : Y \longrightarrow X$ by $h(L_k) :\equiv x_k$ for each k. We also define $\alpha : Y \longrightarrow \mathbb{K}$ by $\alpha(y) := \lambda_{T_y}(\{h(y)\})$. The rest is straightforward. $\square$

4. The upper bound

Proposition 4.1. *Let* $0 < \epsilon < 1/4$. *Suppose that* X *is finite, and let* $T \in \epsilon - \mathbf{DP}(X, Y)$. *Then*

$$\overline{B}\left(T, 2\sqrt{\epsilon}\right) \cap \mathbf{WCM}(X, Y) \neq \emptyset.$$

Proof. We assume that $X = \{x_1, \ldots, x_n\}$. Taking into account Remark 3.1, for each set $C \subset X$, we consider $A_C := E_C \cap \left(\bigcap_{u\in C} E_C^u\right)$, where

$$E_C := \left\{ y \in Y_{2\sqrt{\epsilon}} : \left|\lambda_{T_y}\right|(C) \geq \frac{\|T_y\|}{2} \right\}$$

and

$$E_C^u := \left\{ y \in Y_{2\sqrt{\epsilon}} : \left|\lambda_{T_y}\right|(C \setminus \{u\}) < \frac{\|T_y\|}{2} \right\}.$$

By [1, Lemma 2.1], we know that E_C coincides with the set of all $y \in Y_{2\sqrt{\epsilon}}$ satisfying $\left|\lambda_{T_y}\right|(C) > \|T_y\|/2$, that is,

$$\sum_{x \in C} \left|\left(T\xi_{\{x\}}\right)(y)\right| > \sum_{i=1}^{n} \left|\left(T\xi_{\{x_i\}}\right)(y)\right|/2,$$

and consequently it is both open and closed as a subset of $Y_{2\sqrt{\epsilon}}$. In the same way, each E_C^u is also open and closed in $Y_{2\sqrt{\epsilon}}$, and so is A_C.

Notice that again, by [1, Lemma 2.1], if $y \in E_C$, then

$$\left|\lambda_{T_y}\right|(C) \geq \left(\|T_y\| + \sqrt{\|T_y\|^2 - 4\epsilon} \right) \Big/ 2,$$

and $\left|\lambda_{T_y}\right|(C \setminus \{u\}) \leq \left(\|T_y\| - \sqrt{\|T_y\|^2 - 4\epsilon} \right) \Big/ 2$ for every $y \in E_C^u$. We conclude that $\left|\lambda_{T_y}\right|(\{u\}) \geq \sqrt{\|T_y\|^2 - 4\epsilon}$ for every $y \in A_C$.

On the other hand, it is clear that each $y \in Y_{2\sqrt{\epsilon}}$ belongs to some A_C, so we can make a finite partition of $Y_{2\sqrt{\epsilon}}$ by open and closed sets $B_1, \ldots, B_m$, where each $B_i \subset A_C$ for some set C. This implies that, for each $i = 1, \ldots, m$, there exists a point $u_i \in X$ such that $\left|\lambda_{T_y}(\{u_i\})\right| \geq \sqrt{\|T_y\|^2 - 4\epsilon}$ for every $y \in B_i$. This allows us to define a continuous map $h : Y_{2\sqrt{\epsilon}} \longrightarrow X$ as $h(y) := u_i$ for every $y \in B_i$ (and extend h to the whole of Y by sending $Y \setminus Y_{2\sqrt{\epsilon}}$ to any fixed point $x \in X$). Also take any map $\mathbf{b} : Y \longrightarrow \mathbb{K}$ such that $\mathbf{b}(y) = \lambda_{T_y}(\{h(y)\})$ whenever $y \in Y_{2\sqrt{\epsilon}}$, which is continuous on $Y_{2\sqrt{\epsilon}}$.

We next use the map $\alpha \in C(Y)$ given by

$$\alpha(y) := \sqrt{\frac{\|T_y\| - 2\sqrt{\epsilon}}{\|T_y\| + 2\sqrt{\epsilon}}}$$

for $y \in Y_{2\sqrt{\epsilon}}$, and $\alpha \equiv 0$ on $Y \setminus Y_{2\sqrt{\epsilon}}$, and define a weighted composition map S as

$$(Sf)(y) := \alpha(y)\mathbf{b}(y)f(h(y))$$

for all $f \in C(X)$ and $y \in Y$.

Now, for $y \in Y_{2\sqrt{\epsilon}}$, put $A_y := \mathbf{b}(y)\delta_{h(y)}$. It is easy to check that $\|T_y\| = \|T_y - A_y\| + \|A_y\|$, and that, for $t \in [0, 1]$ and $f \in C(X)$ with $\|f\|_\infty \leq 1$,

$$\begin{aligned} |(Tf)(y) - t\mathbf{b}(y)f(h(y))| &\leq \left|T_y f - A_y f\right| + \left|A_y f - tA_y f\right| \\ &\leq \|T_y - A_y\| + (1-t)\|A_y\| \\ &= \|T_y\| - t\|A_y\| \\ &\leq \|T_y\| - t\sqrt{\|T_y\|^2 - 4\epsilon}. \end{aligned}$$

Obviously, this implies that, given $f \in C(X)$ with $\|f\|_\infty \leq 1$,

$$|(Tf)(y) - (Sf)(y)| \leq \|T_y\| - \alpha(y)\sqrt{\|T_y\|^2 - 4\epsilon} = 2\sqrt{\epsilon}$$

if $y \in Y_{2\sqrt{\epsilon}}$, and $|(Tf)(y) - (Sf)(y)| = |(Tf)(y)| \leq 2\sqrt{\epsilon}$ if $y \notin Y_{2\sqrt{\epsilon}}$. Thus $\|T - S\| \leq 2\sqrt{\epsilon}$. $\square$

Next we provide a result showing that the upper bound $2\sqrt{\epsilon}$ given in Proposition 4.1 is in fact sharp. Notice that this result is valid both for X finite and infinite.

Proposition 4.2. *Let* $0 < \epsilon < 1/4$. *If* Y *is not zero dimensional, then there exists* $T \in \epsilon - \mathbf{DP}(X, Y)$ *such that*

$$B\left(T, 2\sqrt{\epsilon}\right) \cap \mathbf{WCM}(X, Y) = \emptyset.$$

Proof. Consider two points y_1, y_2 in an infinite connected component C of Y, and a function $f_0 \in C(Y)$ such that $-1 \leq f_0 \leq 1$, $f_0(y_1) = -1$, and $f_0(y_2) = 1$. Next take two continuous and even functions $\alpha : [-1, 1] \longrightarrow \left[2\sqrt{\epsilon}, 1\right]$ and $\beta : [-1, 1] \longrightarrow \left[1, 1/\sqrt{1 - 4\epsilon}\right]$, both increasing in $[0, 1]$, such that $\alpha(0) = 2\sqrt{\epsilon}$, $\alpha(1) = 1$, $\beta(0) = 1$, and $\beta(1) = 1/\sqrt{1 - 4\epsilon}$. Taking into account that $x \mapsto x/\sqrt{x^2 - 4\epsilon}$ is decreasing for $x > 2\sqrt{\epsilon}$, we see that $\beta(t)\sqrt{\alpha^2(t) - 4\epsilon} \leq \alpha(t)$ for every $t \in [-1, 1]$.

Now pick two different points $A, B \in X$, and consider $T : C(X) \longrightarrow C(Y)$ defined, for every $f \in C(X)$ and $y \in Y$, by

$$\begin{aligned}(Tf)(y) &:= \frac{\alpha(f_0(y)) + \operatorname{sgn}(f_0(y))\beta(f_0(y))\sqrt{\alpha(f_0(y))^2 - 4\epsilon}}{2} f(A) \\ &\quad + \frac{\alpha(f_0(y)) - \operatorname{sgn}(f_0(y))\beta(f_0(y))\sqrt{\alpha(f_0(y))^2 - 4\epsilon}}{2} f(B),\end{aligned}$$

where sgn denotes the usual sign function.

It is clear that T ϵ-preserves disjointness and has norm 1. Also, since $(T\mathbf{1})(y_i) = 1$ $(i = 1, 2)$, it is easily seen that if a weighted composition map $S = a \cdot f \circ h$ is at a distance less than $2\sqrt{\epsilon}$ from T, then $y_1, y_2 \in c(a)$. On the other hand, if we suppose that $h(y_2) \neq A$, then taking $f_1 \in C(X)$ with $f_1(A) = 1 = \|f_1\|_\infty$ and $f_1(h(y_2)) = 0 = f_1(B)$, we see that

$$|(T - S)(f_1)(y_2)| = 1 > 2\sqrt{\epsilon}.$$

We deduce that, as $\|T - S\| < 2\sqrt{\epsilon}$, then $h(y_2) = A$, and in a similar way $h(y_1) = B$. Since C is connected and $h : c(a) \longrightarrow X$ is continuous, we conclude that there is a point $y_0 \in C$ such that $y_0 \notin c(a)$, that is, $(Sf)(y_0) = 0$ for every $f \in C(X)$. Then it is easy to see that $\|T - S\| \geq \alpha(f_0(y_0)) \geq 2\sqrt{\epsilon}$. $\square$

5. A special case

Proposition 5.1. *Let* $0 < \epsilon < 1/8$. *Suppose that* X *is the one-point compactification of an infinite discrete space. Let* $T \in \epsilon - \mathbf{DP}(X, Y)$. *Then*

$$\overline{B}\left(T, \sqrt{8\epsilon}\right) \cap \mathbf{WCM}(X, Y) \neq \emptyset.$$

Proof. We assume that $X = D \cup \{\infty\}$, where D stands for an infinite discrete space, and put

$$A := \left\{y \in Y : |(T\mathbf{1})(y)| > 2\sqrt{\epsilon}\right\}.$$

Since each T_y ϵ-preserves disjointness, there is, at most, one $x \in D$ with $\left|\left(T\xi_{\{x\}}\right)(y)\right| > \sqrt{\epsilon}$. This allows us to define a map $h : Y \longrightarrow D \cup \{\infty\}$ by $h(y) := x$ if such an x exists, and $h(y) := \infty$ otherwise. Notice that if $y \in A$ and $h(y) \neq x$, then $\left|\left(T\left(\mathbf{1} - \xi_{\{x\}}\right)\right)(y)\right| > \sqrt{\epsilon}$, and, consequently, $\left|\left(T\xi_{\{x\}}\right)(y)\right| < \sqrt{\epsilon}$. Taking this into account, it is easy to see that the map h is continuous on A.

Let us define $\alpha \in C(Y)$ by

$$\alpha(y) := \frac{|(T\mathbf{1})(y)| - 2\sqrt{\epsilon}}{|(T\mathbf{1})(y)|}$$

for all $y \in A$ and $\alpha(y) := 0$ for $y \notin A$. It is clear that $0 \leq \alpha \leq 1$ and $c(\alpha) = A$. We finally define a weighted composition map S by

$$(Sf)(y) := \alpha(y)(T\mathbf{1})(y)f(h(y))$$

for all $f \in C(X)$ and $y \in Y$.

Suppose now that $f \in C(X)$, $\|f\|_\infty \leq 1$. If $2\sqrt{\epsilon} < \left\|T_y\right\| \leq \sqrt{9\epsilon}/2$, then

$$\begin{aligned}|(Tf)(y) - (Sf)(y)| &\leq \left\|T_y\right\| + \alpha(y)\,|(T\mathbf{1})(y)| \\ &\leq 2\sqrt{\frac{9\epsilon}{2}} - 2\sqrt{\epsilon} \\ &< \sqrt{8\epsilon}.\end{aligned}$$

On the other hand, if $\sqrt{9\epsilon}/2 < \left\|T_y\right\|$, then $|(Tf)(y) - (Sf)(y)| \leq \left\|T_y\right\| - \alpha(y)\sqrt{\left\|T_y\right\|^2 - 4\epsilon}$, by [1, Lemma 3.3]. This means in particular that, when $\sqrt{9\epsilon}/2 < \left\|T_y\right\| \leq 8\sqrt{\epsilon}$, then $|(Tf)(y) - (Sf)(y)| \leq 8\sqrt{\epsilon}$. Also, when $\left\|T_y\right\| \geq 8\sqrt{\epsilon}$, by [1, Corollary 2.6], $|(T\mathbf{1})(y)| \geq \sqrt{\left\|T_y\right\|^2 - 4\epsilon} \geq 2\sqrt{\epsilon}$, which implies that

$$\left\|T_y\right\| - \alpha(y)\sqrt{\left\|T_y\right\|^2 - 4\epsilon} \leq \left\|T_y\right\| - \sqrt{\left\|T_y\right\|^2 - 4\epsilon} + (1 - \alpha(y))\,|(T\mathbf{1})(y)|.$$

Using that the map $\gamma(t)$ given in Lemma 3.4 is decreasing, and evaluating it at $t = \sqrt{8\epsilon}$, we deduce that, for $\left\|T_y\right\| > \sqrt{8\epsilon}$,

$$\begin{aligned}\left\|T_y\right\| - \alpha(y)\sqrt{\left\|T_y\right\|^2 - 4\epsilon} &\leq \sqrt{8\epsilon} - 2\sqrt{\epsilon} + (1 - \alpha(y))\,|(T\mathbf{1})(y)| \\ &= \sqrt{8\epsilon}.\end{aligned}$$

The fact that $\|T - S\| \leq \sqrt{8\epsilon}$ follows easily. $\square$

The following example shows that the bound in Proposition 5.1 (hence, in Theorem 2.3) is sharp. Its conclusions also hold (with slight changes in the proof) using the same Y and any X containing a nonconstant convergent sequence.

Example 5.2. Here we construct a space Y such that, for the one-point compactification $X = D \cup \{\infty\}$ of any infinite discrete space D and any $\epsilon \in (0, 1/8)$, there exists a norm one operator that ϵ-preserves disjointness whose distance to any weighted composition map is at least $\sqrt{8\epsilon}$.

Given $r > 0$, we denote by $C(r)$ the circle with center 0 and radius r in the complex plane. We take a strictly decreasing sequence (r_n) in $\mathbb{R}$ converging to 0 and the interval $[-r_1, 0]$, and define $Y \subset \mathbb{C}$ by

$$Y := [-r_1, 0] \cup \bigcup_{n=1}^{\infty} C(r_n).$$

Next we construct a norm one operator 1/8-preserving disjointness. Let

$$\pi_0 := \frac{1}{2} - \frac{\sqrt{2}}{4},$$

and consider a continuous map $\alpha : \bigcup_{n=1}^{\infty} C(r_n) \longrightarrow [0, \pi_0]$ such that $\alpha(-r_n) = 0$ and $\alpha(r_n) = \pi_0$ for every $n \in \mathbb{N}$.

Since D is infinite, we may assume that $\mathbb{N} \subset D$. For each $f \in C(X)$ and $n \in \mathbb{N}$, we define, for $z \in C(r_n)$,

$$(Tf)(z) := \left(\alpha(z) + \sqrt{2}/2\right) f(2n) - \alpha(z) f(2n-1).$$

On the other hand, if $n \in \mathbb{N}$, then each $z \in (-r_n, -r_{n+1})$ is of the form

$$z = -\left(t r_n + (1-t) r_{n+1}\right),$$

where t belongs to the open interval (0, 1). In this case, we define

$$\begin{aligned}(Tf)(z) &:= t\,(Tf)\,(-r_n) + (1-t)\,(Tf)\,(-r_{n+1}) \\ &= \frac{\sqrt{2}}{2}\left[t f(2n) + (1-t) f(2n+2)\right].\end{aligned}$$

Finally we put

$$(Tf)(0) := \frac{\sqrt{2}}{2} f(\infty).$$

It is apparent that $T : C(X) \longrightarrow C(Y)$ is linear and continuous, with $\|T\| = 1$. Furthermore it is easy to see that if $f, g \in C(X)$ satisfy $\|f\|_\infty = 1 = \|g\|_\infty$ and $fg = 0$, then $|(Tf)(z)(Tg)(z)| \leq 1/8$ for every $z \in Y$, that is, T 1/8-preserves disjointness.

We will now check that we cannot find a weighted composition map "near" T. Namely, if $S : C(X) \longrightarrow C(Y)$ denotes a weighted composition map, then we claim that $\|S - T\| \geq 1$.

Consider the continuous map $h : c(S\mathbf{1}) \longrightarrow X$ given by S. If $r_n \notin c(S\mathbf{1})$ for some $n \in \mathbb{N}$, then we take $f_n := \xi_{\{2n\}} - \xi_{\{2n-1\}}$. It is clear that $\|f_n\|_\infty = 1$, $(Tf_n)(r_n) = 1$ and, as $r_n \notin c(S\mathbf{1})$, then $(Sf_n)(r_n) = 0$. As a consequence $\|S - T\| \geq 1$. It is also easy to see that we obtain the same conclusion if $h(r_n) \notin \{2n, 2n-1\}$. So we assume that $r_n \in c(S\mathbf{1})$ and $h(r_n) \in \{2n, 2n-1\}$ for every $n \in \mathbb{N}$.

Now, if we suppose that $0 \notin c(S\mathbf{1})$, then $(S\mathbf{1})(0) = 0$. Therefore, given any $\delta > 0$, there exists a neighborhood U of 0 in Y such that $|(S\mathbf{1})(z)| < \delta$ for all $z \in U$. Choose now $r_n \in U$, and let f_n be as above. It is apparent that either $(\mathbf{1} - f_n)(h(r_n)) = 0$ or $(\mathbf{1} + f_n)(h(r_n)) = 0$, which implies that $(S\mathbf{1})(r_n) - (Sf_n)(r_n) = 0$ or $(S\mathbf{1})(r_n) + (Sf_n)(r_n) = 0$. Consequently, $|(Sf_n)(r_n)| = |(S\mathbf{1})(r_n)| < \delta$ and, as in the previous cases, we easily deduce that $\|S - T\| \geq 1 - \delta$. Therefore $\|S - T\| \geq 1$.

Finally, if we suppose that $0 \in c(S\mathbf{1})$, then there exists $s > 0$ such that $B(0, s) \cap Y \subset c(S\mathbf{1})$. Also h is continuous and $B(0, s) \cap Y$ is connected, so $h\,(B(0, s) \cap Y)$ is constant. This is obviously impossible by our assumptions on $h(r_n)$. Hence, we obtain $\|S - T\| \geq 1$.

Let $0 < \epsilon < 1/8$. We are going to construct a norm one T' which ϵ-preserves disjointness such that, for all weighted composition maps S', $\|T' - S'\| \geq \sqrt{8\epsilon}$. Let

$$\lambda := \sqrt{8\epsilon},$$

and let $X' := X \cup \{\mathbf{0}\}$ (where $\mathbf{0}$ is an isolated point in X', $\mathbf{0} \notin X$) and $Y' := Y \cup \{2r_1\} \subset \mathbb{C}$. Define a linear map $T' : C(X') \longrightarrow C(Y')$ by $(T'f)(2r_1) := f(\mathbf{0})$ and, for all $z \in Y$, $(T'f)(z) := \lambda(Tf_r)(z)$, where f_r is the restriction of f to X.

Since T $1/8$-preserves disjointness and $\epsilon = \lambda^2/8$, then T' ϵ-preserves disjointness. The conclusion follows as in [1, Example 4.4].

6. Proofs of the main results

Proof of Theorem 2.1. If $\epsilon < 1/4$, then $\mathbf{S}(X, Y)(\epsilon) \geq 2\sqrt{\epsilon}$ by Corollary 3.2. Since, for fixed X and Y, the function $\mathbf{S}(X, Y)(\epsilon)$ is increasing in ϵ, $\mathbf{S}(X, Y)(\epsilon) \geq 1$ for $\epsilon \geq 1/4$, and we obtain the inequality $\min\{2\sqrt{\epsilon}, 1\} \leq \mathbf{S}(X, Y)(\epsilon)$. On the other hand, we have $\mathbf{S}(X, Y)(\epsilon) \leq \sqrt{17\epsilon/2}$ when $\epsilon < \sqrt{17/2}$ (see [1, Theorem 1.1]). Since $\mathbf{S}(X, Y)(\epsilon) \leq 1$, we easily infer the other inequality. $\square$

The same ideas given in the proof of Theorem 2.1 must also be followed in the following.

Proof of Theorem 2.2. Easy by Proposition 3.5 and Corollary 3.2. $\square$

Proof of Theorem 2.3. Easy by Proposition 5.1. $\square$

Proof of Theorem 2.4. The result follows from Propositions 4.1 and 4.2, and Corollaries 3.3 and 3.9. $\square$

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